

A MULTILEVEL CORRECTION SCHEME FOR NONSYMMETRIC EIGENVALUE PROBLEMS BY FINITE ELEMENT METHODS

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Abstract. A multilevel correction scheme is proposed to solve defective and nondefective of nonsymmetric partial differential operators by the finite element method. The method includes multi correction steps in a sequence of finite element spaces. In each correction step, we only need to solve two source problems on a finer finite element space and two eigenvalue problems on the coarsest finite element space. The accuracy of the eigenpair approximation is improved after each correction step. This correction scheme improves overall efficiency of the finite element method in solving nonsymmetric eigenvalue problems.

Key words. Nonsymmetric eigenvalue problem, multilevel correction, finite element method, high-efficiency

AMS subject classifications. 65N30, 65N25, 65L15, 65B99

1. Introduction. As we know, the numerical approximation of eigenvalue problems plays a central role in the analysis of the stability for nonlinear partial differential equations. For example in fluid mechanics, the analysis of the hydrodynamic stability always leads to a nonsymmetric eigenvalue problems (see [1, 7, 8]). The stability of the underlying flow depends on the real part of the eigenvalue which has the smallest real part (see [1, 8]). For more details, please refer [1, 8, 15]. The aim of understanding the stability of nonlinear partial differential equations naturally leads to the computation of the eigenvalue problems with some numerical methods. The main content of this paper is to design an efficient finite element method to compute nonsymmetric eigenvalue problems.

Recently, a multigrid method is designed to solve the self-adjoint eigenvalue problem based on a type of multilevel correction method [10, 11, 12, 18]. But as we know, the analysis of the stability for nonlinear partial differential equations always leads to nonsymmetric eigenvalue problems [1, 8] and the extensions of the multilevel method for self-adjoint eigenvalue problems to the nonsymmetric ones is not direct [9, 23, 25] and needs more analysis. So the purpose of this paper is to propose a multilevel correction scheme to solve nonsymmetric eigenvalue problems based on the finite element method. In the past, a two-grid finite element method was proposed and analyzed by Xu and Zhou in [23] for symmetric eigenvalue problems. Latter, Kolman used this idea to design a two-level method for nonsymmetric eigenvalue problems in [9]. Yang and Fan [25] also studied a two-grid method for nonsymmetric eigenvalue problems. As an alternative approach, in [13, 14, 17], the authors used a recovery technique PPR to improve the convergence rate for both symmetric and nonsymmetric eigenvalue problems. All these methods are designed for the nonsymmetric eigenvalue problems under the assumption that the ascent of the concerned eigenvalues is only one which means the algebraic eigenspace is the same as the geometric eigenspace.

Along the line of multilevel correction method, here we present a multilevel correction scheme to solve nonsymmetric eigenvalue problems without the ascent assumption. With the proposed method, solving nonsymmetric eigenvalue problems will not

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be much more expensive than solving corresponding source problems. The correction method for eigenvalue problems in this paper is based on a series of finite element spaces with different levels of accuracy which are related to the multilevel method (cf. [20]).

The standard Galerkin finite element method for nonsymmetric eigenvalue problems has been extensively investigated, e.g. Babuška and Osborn [2, 3], Chatelin [5] and references cited therein. Here we adopt some basic results in these papers to carry on error estimates for our multilevel correction scheme. It will be shown that the convergence rate of the eigenpair approximations can be improved after each correction step.

Our multilevel correction procedure can be described as follows: (1) solve an eigenvalue problem in the coarsest finite element space; (2) solve a source problem in an augmented space with the associated eigenfunction from (1) as the load vector; (3) solve the eigenvalue problem again on a finite element space constructed by enhancing the coarsest finite element space with the eigenfunction obtained in step (2). Then go to step (2) for the next loop.

In this method, we replace solving the eigenvalue problem in finer finite element spaces by solving a series of boundary value problems in a series of nested finite element spaces (with the finest space as the last one) and a series of eigenvalue problems in the coarsest finite element space; and yet, we achieve the same accuracy as solving the eigenvalue problem in the finest space. It is well known that there exist multigrid methods that solve boundary value problems with the optimal computational work (cf. [21]). Therefore, combined with the multigrid method, our correction method improves overall efficiency in solving nonsymmetric eigenvalue problems (cf. [18, 19]).

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for nonsymmetric eigenvalue problems. An one level correction scheme is described and analyzed in Section 3. In Section 4, we propose and analyze a multilevel correction algorithm to solve nonsymmetric eigenvalue problems by the finite element method. Some numerical examples are presented in Section 5 to validate our theoretical analysis and some concluding remarks are given in the last section.

2. Discretization by finite element method. In this section, we introduce some notation and error estimates of the finite element approximation for nonsymmetric eigenvalue problems. Throughout this paper, the letter C (with or without subscripts) denotes a generic positive constant which may be different at different occurrences. For convenience, we use symbols \lesssim , \gtrsim , and \approx , such that $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$, and $x_3 \approx y_3$ have meanings: $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$, and $c_3 x_3 \leq y_3 \leq C_3 x_3$, for some constants C_1, c_2, c_3 , and C_3 that are independent of mesh sizes (cf. [20]).

We consider the following eigenvalue problem:

Find $\lambda \in \mathcal{C}$ and u such that

$$\begin{cases} -\nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + \phi u &= \lambda \varphi u, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \\ \int_{\Omega} \varphi |u|^2 d\Omega &= 1, \end{cases} \quad (2.1)$$

where $\Omega \subset \mathcal{R}^d$ is a bounded polygonal domain, $A \in \mathcal{C}^{d \times d}$, $\mathbf{b} \in \mathcal{C}^d$, ϕ is a function defined on Ω and φ is a real positive function with $\varphi \geq c_0 > 0$.

We define $V := H_0^1(\Omega)$ with the usual norm $\|\cdot\|_1$. The corresponding variational form of (2.1) can be stated as follows:

Find $(\lambda, u) \in \mathcal{C} \times V$ such that $b(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v), \quad \forall v \in V, \quad (2.2)$$

where

$$\begin{aligned} a(u, v) &= (A\nabla u, \overline{\nabla v}) + (\mathbf{b} \cdot \nabla u, \bar{v}) + (\phi u, \bar{v}), \\ b(u, v) &= (\varphi u, \bar{v}) \end{aligned}$$

with (\cdot, \cdot) denoting the inner product in the space $L^2(\Omega)$. The corresponding adjoint eigenvalue problem is:

Find $(\lambda, u^*) \in \mathcal{C} \times V$ such that $b(u^*, u^*) = 1$ and

$$a(v, u^*) = \lambda b(v, u^*), \quad \forall v \in V. \quad (2.3)$$

In the sequel, we also use the norm $\|v\|_b = \sqrt{b(v, v)}$ which is equivalent to the $L^2(\Omega)$ norm $\|\cdot\|_0$. Here the bilinear form $a(\cdot, \cdot)$ is assumed to satisfy

$$\|w\|_1 \lesssim \sup_{v \in V} \frac{a(w, v)}{\|v\|_1} \quad \text{and} \quad \|w\|_1 \lesssim \sup_{v \in V} \frac{a(v, w)}{\|v\|_1}, \quad \forall w \in V. \quad (2.4)$$

We further assume that $a(\cdot, \cdot)$ is V -elliptic, i.e.,

$$\|u\|_1^2 \lesssim \operatorname{Re} a(u, u), \quad \forall u \in V. \quad (2.5)$$

2.1. Operator reformulation. We introduce the operators $T, T_* \in \mathcal{L}(V)$ defined by the equation

$$a(Tu, v) = b(u, v) = a(u, T_*v), \quad \forall u, v \in V. \quad (2.6)$$

The eigenvalue problem (2.2) can be written as an operator form for $\lambda \neq 0$ (denoting $\mu := \lambda^{-1}$):

$$Tu = \mu u, \quad (2.7)$$

with

$$T_*u^* = \bar{\mu}u^* \quad (2.8)$$

for the adjoint eigenvalue problem (2.3). Note that ellipticity condition (2.5) guarantees that every eigenvalue λ is nonzero. It is well known that the operators T and T_* are compact. Thus the spectral theory for compact operators gives us a complete characterization of the eigenvalue problem (2.2).

There is a countable set of eigenvalues of (2.2). Let λ be an eigenvalue of problem (2.2). There exists a smallest integer α which are called the ascent such that

$$N((T - \mu)^\alpha) = N((T - \mu)^{\alpha+1}), \quad (2.9)$$

where N denotes the null space and we use the notation $\mu = \lambda^{-1}$. Let $M(\lambda) = M_{\lambda, \mu} = N((T - \mu)^\alpha)$ and $Q(\lambda) = Q_{\lambda, \mu} = N(T - \mu)$ denote the algebraic and geometric eigenspaces, respectively. The subspaces $Q(\lambda) \subset M(\lambda)$ are finite dimensional. The numbers $m = \dim M(\lambda)$ and $q = \dim Q(\lambda)$ are called the algebraic and the geometric multiplicities of μ (and λ). The vectors in $M(\lambda)$ are generalized eigenvectors. The order of a generalized eigenvector is the smallest integer j such that $(T - \mu)^j u = 0$ (vectors in $Q(\lambda)$ being generalized eigenvectors of order 1). Let us point out that a generalized eigenvector u^j of order j satisfies

$$a(u^j, v) = \lambda b(u^j, v) + \lambda a(u^{j-1}, v), \quad \forall v \in V, \quad (2.10)$$

where u^{j-1} is a generalized eigenvector of order $j - 1$.

Similarly we define the spaces of (generalized) eigenvectors for the adjoint problem

$$M^*(\lambda) = M_{\lambda, \mu}^* = N((T_* - \bar{\mu})^\alpha) \quad \text{and} \quad Q^*(\lambda) = Q_{\lambda, \mu}^* = N(T_* - \bar{\mu}).$$

Note that μ is an eigenvalue of T (λ is an eigenvalue of problem (2.2)) if and only if $\bar{\mu}$ is an eigenvalue of T_* (λ is an eigenvalue of adjoint problem (2.3)) with the ascent α and the algebraic multiplicity m for both eigenvalues being the same.

2.2. Galerkin discretization. Now, let us define the finite element approximations for the problem (2.2). First we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K . The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh \mathcal{T}_h , we construct a finite element space denoted by $V_h \subset V$. In order to define our multilevel correction method, we start the process on an initial mesh \mathcal{T}_H with mesh size H and the initial finite element space V_H defined on \mathcal{T}_H . In this paper, the finite element space V_h is assumed to satisfy

$$\|w_h\|_1 \lesssim \sup_{v_h \in V_h} \frac{a(w_h, v_h)}{\|v_h\|_1} \quad \text{and} \quad \|w_h\|_1 \lesssim \sup_{v_h \in V_h} \frac{a(v_h, w_h)}{\|v_h\|_1}, \quad \forall w_h \in V_h. \quad (2.11)$$

The standard Galerkin discretization of the problem (2.2) is the following:

Find $(\lambda_h, u_h) \in \mathcal{C} \times V_h$ such that $b(u_h, u_h) = 1$ and

$$a(u_h, v_h) = \lambda_h b(u_h, v_h), \quad \forall v_h \in V_h. \quad (2.12)$$

Similarly, the discretization of the adjoint problem (2.3) can be defined as:

Find $(\lambda_h, u_h^*) \in \mathcal{C} \times V_h$ such that $b(u_h^*, u_h^*) = 1$ and

$$a(v_h, u_h^*) = \lambda_h b(v_h, u_h^*), \quad \forall v_h \in V_h. \quad (2.13)$$

By introducing Galerkin projections $P_h, P_h^* \in \mathcal{L}(V, V_h)$ with the following equations

$$\begin{aligned} a(P_h u, v_h) &= a(u, v_h), & \forall u \in V, \forall v_h \in V_h, \\ a(v_h, u) &= a(v_h, P_h^* u), & \forall u \in V, \forall v_h \in V_h, \end{aligned}$$

the equation (2.12) can be rewritten as an operator form with $\mu_h := \lambda_h^{-1}$ (Note that P_h is a bounded operator),

$$P_h T u_h = \mu_h u_h. \quad (2.14)$$

Similarly for the adjoint problem (2.13), we have

$$P_h^* T_* u_h^* = \bar{\mu}_h u_h^*. \quad (2.15)$$

2.3. Spectral approximation of compact operators. Let μ be an eigenvalue (with algebraic multiplicity m) of the compact operator T . If T is approximated by a sequence of compact operators T_h converging to T in norm, i.e., $\lim_{h \rightarrow 0+} \|T - T_h\|_1 = 0$, then for h sufficiently small μ is approximated by m numerical eigenvalues $\{\mu_{j,h}\}_{j=1,\dots,m}$ (counted according to their algebraic multiplicities) of T_h , i.e.,

$$\lim_{h \rightarrow 0+} \mu_{j,h} = \mu \quad \text{for } j = 1, \dots, m.$$

The space of generalized eigenvectors of T is approximated by the subspace

$$M_h(\lambda) = M_h^{\lambda, \mu} = \sum_{j=1}^m N((T_h - \mu_{j,h})^{\alpha_{\mu_{j,h}}}), \quad (2.16)$$

where $\alpha_{\mu_{j,h}}$ is the smallest integer such that $N((T_h - \mu_{j,h})^{\alpha_{\mu_{j,h}}}) = N((T_h - \mu_{j,h})^{\alpha_{\mu_{j,h}}+1})$. We similarly define the space $Q_h(\lambda) = Q_h^{\lambda, \mu} = \sum_{j=1}^m N(T_h - \mu_{j,h})$ and counterparts $M_h^*(\lambda)$, $Q_h^*(\lambda)$ for the adjoint problem.

Now, we describe a computational scheme to produce the algebraic eigenspace $M_h(\lambda)$ from the geometric eigenspace $Q_h(\lambda) = \{u_{1,h}, \dots, u_{q,h}\}$ corresponding to eigenvalues $\{\lambda_{1,h}, \dots, \lambda_{q,h}\}$, which converge to the same eigenvalue λ .

Starting from all eigenfunctions in the geometric eigenspace $Q_h(\lambda)$ (of order 1), we use the following recursive process to compute algebraic eigenspaces (cf. [16])

$$\begin{cases} a(u_{j,h}^\ell, v_h) - \lambda_{j,h} b(u_{j,h}^\ell, v_h) &= \lambda_{j,h} a(u_{j,h}^{\ell-1}, v_h), \quad \forall v_h \in V_h, \\ b(u_{j,h}^\ell, v_h) &= 0, \quad \forall v_h \in Q_h(\lambda), \end{cases} \quad (2.17)$$

where $\ell \geq 2$, $u_{j,h}^\ell$ is the general eigenfunction of order ℓ and $u_{j,h}^1 = u_{j,h} \in Q_h(\lambda)$ for $j = 1, \dots, q$.

With the above process, we generate the algebraic eigenspace

$$M_h(\lambda) = \{u_{1,h}, \dots, u_{q,h}, \dots, u_{m,h}\}$$

corresponding to eigenvalues $\{\lambda_{1,h}, \dots, \lambda_{q,h}, \dots, \lambda_{m,h}\}$, which converge to the same eigenvalue λ . Similarly, we can produce the adjoint algebraic eigenspace $M_h^*(\lambda)$ from the geometric eigenspace $Q_h^*(\lambda)$.

For two linear spaces A and B , we denote

$$\widehat{\Theta}(A, B) = \sup_{w \in A, \|w\|_1=1} \inf_{v \in B} \|w - v\|_1, \quad \widehat{\Phi}(A, B) = \sup_{w \in A, \|w\|_b=1} \inf_{v \in B} \|w - v\|_b,$$

and define gaps between A and B in $\|\cdot\|_1$ as

$$\Theta(A, B) = \max \{\widehat{\Theta}(A, B), \widehat{\Theta}(B, A)\}, \quad (2.18)$$

and in $\|\cdot\|_b$ as

$$\Phi(A, B) = \max \{\widehat{\Phi}(A, B), \widehat{\Phi}(B, A)\}. \quad (2.19)$$

Before introducing the convergence results of the finite element approximation for nonsymmetric eigenvalue problems, we define the following notation

$$\delta_h(\lambda) = \sup_{u \in M(\lambda), \|u\|_1=1} \inf_{v_h \in V_h} \|u - v_h\|_1, \quad (2.20)$$

$$\delta_h^*(\lambda) = \sup_{u^* \in M^*(\lambda), \|u^*\|_1=1} \inf_{v_h \in V_h} \|u^* - v_h\|_1, \quad (2.21)$$

$$\rho_h(\lambda) = \sup_{u \in M(\lambda), \|u\|_b=1} \inf_{v_h \in V_h} \|u - v_h\|_b, \quad (2.22)$$

$$\rho_h^*(\lambda) = \sup_{u^* \in M^*(\lambda), \|u^*\|_b=1} \inf_{v_h \in V_h} \|u^* - v_h\|_b, \quad (2.23)$$

$$\eta_a(h) = \sup_{f \in V, \|f\|_b=1} \inf_{v \in V_h} \|Tf - v\|_1, \quad (2.24)$$

$$\eta_a^*(h) = \sup_{f \in V, \|f\|_b=1} \inf_{v \in V_h} \|T_* f - v\|_1. \quad (2.25)$$

In order to derive error bounds for eigenpair approximations in the weak norm $\|\cdot\|_b$, we need the following error estimates in the weak norm $\|\cdot\|_b$ of the finite element approximation.

LEMMA 2.1. ([3, Lemma 3.3 and Lemma 3.4])

$$\eta_a(h) = o(1), \quad \eta_a^*(h) = o(1) \quad \text{as } h \rightarrow 0, \quad (2.26)$$

and

$$\rho_h(\lambda) \lesssim \eta_a^*(h) \delta_h(\lambda), \quad (2.27)$$

$$\rho_h^*(\lambda) \lesssim \eta_a(h) \delta_h^*(\lambda). \quad (2.28)$$

The following theorem is a basic tool for our error estimates.

THEOREM 2.2. ([3, Section 8]) When the mesh size h is small enough, we have

$$\Theta(M(\lambda), M_h(\lambda)) \lesssim \delta_h(\lambda), \quad \Theta(M^*(\lambda), M_h^*(\lambda)) \lesssim \delta_h^*(\lambda), \quad (2.29)$$

$$\Phi(M(\lambda), M_h(\lambda)) \lesssim \rho_h(\lambda), \quad \Phi(M^*(\lambda), M_h^*(\lambda)) \lesssim \rho_h^*(\lambda), \quad (2.30)$$

$$|\lambda - \hat{\lambda}_h| \lesssim \delta_h(\lambda) \delta_h^*(\lambda), \quad (2.31)$$

where $\hat{\lambda}_h = \frac{1}{m} \sum_{j=1}^m \lambda_{j,h}$ with $\lambda_{1,h}, \dots, \lambda_{m,h}$ converging to λ .

3. One correction step. In this section, we present an one-step correction procedure to improve the accuracy of the current eigenvalue and eigenfunction approximations. This correction method contains solving some auxiliary source problems in a finer finite element space and two eigenvalue problems on a coarse finite element space.

Assume that we have obtained the algebraic eigenpair approximations $(\lambda_{j,h_k}, u_{j,h_k}) \in \mathcal{R} \times V_{h_k}$ and the corresponding adjoint ones $(\lambda_{j,h_k}, u_{j,h_k}^*) \in \mathcal{R} \times V_{h_k}$ for $j = i, \dots, i+m-1$, where eigenvalues $\{\lambda_{j,h_k}\}_{j=i}^{i+m-1}$ converge to the desired eigenvalue λ_i of (2.2). Now we introduce a correction step to improve the accuracy of the current eigenpair approximations. Let $V_{h_{k+1}} \subset V$ be the conforming finite element space based on a finer mesh $\mathcal{T}_{h_{k+1}}$ which is produced by refining \mathcal{T}_{h_k} in the regular way. We start from a conforming linear finite element space V_H on the coarsest mesh \mathcal{T}_H to design the following one correction step.

ALGORITHM 3.1. *One Correction Step*

1. For $j = i, \dots, i+m-1$ Do

Solve the following two boundary value problems:

Find $\tilde{u}_{j,h_{k+1}} \in V_{h_{k+1}}$ such that

$$a(\tilde{u}_{j,h_{k+1}}, v_{h_{k+1}}) = b(u_{j,h_k}, v_{h_{k+1}}), \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}. \quad (3.1)$$

Find $\tilde{u}_{j,h_{k+1}}^* \in V_{h_{k+1}}$ such that

$$a(v_{h_{k+1}}, \tilde{u}_{j,h_{k+1}}^*) = b(v_{h_{k+1}}, u_{j,h_k}^*), \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}. \quad (3.2)$$

End Do

2. Define two new finite element spaces

$$V_{H,h_{k+1}} = V_H \oplus \text{span}\{\tilde{u}_{i,h_{k+1}}, \dots, \tilde{u}_{i+m-1,h_{k+1}}\}$$

and

$$V_{H,h_{k+1}}^* = V_H \oplus \text{span}\{\tilde{u}_{i,h_{k+1}}^*, \dots, \tilde{u}_{i+m-1,h_{k+1}}^*\}.$$

Solve the following two eigenvalue problems:

Find $(\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}) \in \mathcal{R} \times V_{H,h_{k+1}}$ such that $b(u_{j,h_{k+1}}, u_{j,h_{k+1}}) = 1$ and

$$a(u_{j,h_{k+1}}, v_{H,h_{k+1}}) = \lambda_{j,h_{k+1}} b(u_{j,h_{k+1}}, v_{H,h_{k+1}}), \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}^*. \quad (3.3)$$

Find $(\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}^*) \in \mathcal{R} \times V_{H,h_{k+1}}^*$ such that $b(u_{j,h_{k+1}}^*, u_{j,h_{k+1}}^*) = 1$ and

$$a(v_{H,h_{k+1}}, u_{j,h_{k+1}}^*) = \lambda_{j,h_{k+1}} b(v_{H,h_{k+1}}, u_{j,h_{k+1}}^*), \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}. \quad (3.4)$$

3. Choose $2q$ eigenpairs $\{\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}\}_{j=i}^{i+q-1}$ and $\{\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}^*\}_{j=i}^{i+q-1}$ to define two new geometric eigenspaces

$$Q_{h_{k+1}}(\lambda_i) = \text{span}\{u_{i,h_{k+1}}, \dots, u_{i+q-1,h_{k+1}}\}$$

and

$$Q_{h_{k+1}}^*(\lambda_i) = \text{span}\{u_{i,h_{k+1}}^*, \dots, u_{i+q-1,h_{k+1}}^*\}.$$

Based on these two geometric eigenspaces, compute the corresponding algebraic eigenspaces

$$M_{h_{k+1}}(\lambda_i) = \text{span}\{u_{i,h_{k+1}}, \dots, u_{i+m-1,h_{k+1}}\} \quad (3.5)$$

and

$$M_{h_{k+1}}^*(\lambda_i) = \text{span}\{u_{i,h_{k+1}}, \dots, u_{i+m-1,h_{k+1}}\}. \quad (3.6)$$

The final output is:

$$\begin{aligned} & (\{\lambda_{j,h_{k+1}}\}_{j=i}^{i+m-1}, M_{h_{k+1}}(\lambda_i), M_{h_{k+1}}^*(\lambda_i)) = \\ & \text{Correction}(V_H, \{\lambda_{j,h_k}\}_{j=i}^{i+m-1}, M_{h_k}(\lambda_i), M_{h_k}^*(\lambda_i), V_{h_{k+1}}). \end{aligned}$$

REMARK 3.1. Since in Step 1 of Algorithm 3.1, the solving process for the boundary value problems is independent of each other for different j , we can implement them in parallel. Furthermore, the designing for this algorithm does not need the ascent assumption as in [9, 25].

THEOREM 3.1. Assume there exist real numbers $\varepsilon_{h_k}(\lambda_i)$ and $\varepsilon_{h_k}^*(\lambda_i)$ such that the given eigenpairs $(\{\lambda_{j,h_k}\}_{j=i}^{i+m-1}, M_{h_k}(\lambda_i), M_{h_k}^*(\lambda_i))$ in One Correction Step 3.1 have following error estimates

$$\Theta(M(\lambda_i), M_{h_k}(\lambda_i)) \lesssim \varepsilon_{h_k}(\lambda_i), \quad (3.7)$$

$$\Theta(M^*(\lambda_i), M_{h_k}^*(\lambda_i)) \lesssim \varepsilon_{h_k}^*(\lambda_i), \quad (3.8)$$

$$\Phi(M(\lambda_i), M_{h_k}(\lambda_i)) \lesssim \eta_a^*(H) \varepsilon_{h_k}(\lambda_i), \quad (3.9)$$

$$\Phi(M^*(\lambda_i), M_{h_k}^*(\lambda_i)) \lesssim \eta_a(H) \varepsilon_{h_k}^*(\lambda_i). \quad (3.10)$$

Then after one correction step, the resultant eigenpair approximation $(\{\lambda_{j,h_{k+1}}\}_{j=i}^{i+m-1}, M_{h_{k+1}}(\lambda_i), M_{h_{k+1}}^*(\lambda_i))$ have following error estimates

$$\Theta(M(\lambda_i), M_{h_{k+1}}(\lambda_i)) \lesssim \varepsilon_{h_{k+1}}(\lambda_i), \quad (3.11)$$

$$\Theta(M^*(\lambda_i), M_{h_{k+1}}^*(\lambda_i)) \lesssim \varepsilon_{h_{k+1}}^*(\lambda_i), \quad (3.12)$$

$$\Phi(M(\lambda_i), M_{h_{k+1}}(\lambda_i)) \lesssim \eta_a^*(H) \varepsilon_{h_{k+1}}(\lambda_i), \quad (3.13)$$

$$\Phi(M^*(\lambda_i), M_{h_{k+1}}^*(\lambda_i)) \lesssim \eta_a(H) \varepsilon_{h_{k+1}}^*(\lambda_i), \quad (3.14)$$

where $\varepsilon_{h_{k+1}}(\lambda_i) := \eta_a^*(H) \varepsilon_{h_k}(\lambda_i) + \delta_{h_{k+1}}(\lambda_i)$ and $\varepsilon_{h_{k+1}}^*(\lambda_i) := \eta_a(H) \varepsilon_{h_k}^*(\lambda_i) + \delta_{h_{k+1}}^*(\lambda_i)$.

Proof. From (2.10), there exist the basis functions $\{u_j\}_{j=i}^{i+m-1}$ of $M(\lambda_i)$ such that

$$a(u_j, v) = b\left(\sum_{k=i}^{i+m-1} p_{jk}(\lambda_i) u_k, v\right), \quad \forall v \in V, \quad (3.15)$$

where $p_{jk}(\cdot)$ denotes a polynomial of degree no more than α for $k = i, \dots, j$ with $p_{jj}(\lambda_i) = \lambda_i$ and $p_{jk}(\lambda_i) = 0$ for $j < k \leq i + m - 1$. We can define a matrix $\mathcal{P} := (p_{j+1-i, k+1-i})_{i \leq j, k \leq i+m-1} \in \mathcal{C}^{q \times q}$ such that

$$a(U, v) = b(\mathcal{P}U, v), \quad \forall v \in V, \quad (3.16)$$

where $U := (u_i, \dots, u_{i+m-1})^T$. It is easy to know that the matrix \mathcal{P} is nonsingular providing $\lambda_i \neq 0$.

For each $\tilde{u}_{j,h_{k+1}}$, from the definitions of $\Theta(M(\lambda_i), M_{h_k}(\lambda_i))$ and $\Phi(M(\lambda_i), M_{h_k}(\lambda_i))$, there exist a vector $\mathcal{R}_j := (c_1, \dots, c_m)^T \in \mathcal{C}^{m \times 1}$ such that

$$\|u_{j,h_k} - \mathcal{R}_j^T U\|_1 \lesssim \varepsilon_{h_k}(\lambda_i), \quad \text{for } j = i, \dots, i + m - 1, \quad (3.17)$$

$$\|u_{j,h_k} - \mathcal{R}_j^T U\|_0 \lesssim \eta_a^*(H) \varepsilon_{h_k}(\lambda_i), \quad \text{for } j = i, \dots, i + m - 1. \quad (3.18)$$

For any $v_{h_{k+1}} \in V_{h_{k+1}}$, we have

$$\begin{aligned} & |a(\tilde{u}_{j,h_{k+1}} - P_{h_{k+1}} \mathcal{R}_j^T \mathcal{P}^{-1} U, v_{h_{k+1}})| = |a(\tilde{u}_{j,h_{k+1}} - \mathcal{R}_j^T \mathcal{P}^{-1} U, v_{h_{k+1}})| \\ & = b(u_j^{h_k} - \mathcal{R}_j^T \mathcal{P}^{-1} \mathcal{P} U, v_{h_{k+1}}) = |b(u_j^{h_k} - \mathcal{R}_j^T U, v_{h_{k+1}})| \\ & \lesssim \eta_a^*(H) \varepsilon_{h_k}(\lambda_i) \|v_{h_{k+1}}\|_1, \quad \text{for } j = i, \dots, i + m - 1. \end{aligned} \quad (3.19)$$

From (2.11) and (3.19), the following estimate holds

$$\begin{aligned} \|\tilde{U}_{j,h_{k+1}} - P_{h_{k+1}} \mathcal{R}_j^T \mathcal{P}^{-1} U\|_1 & \lesssim \eta_a^*(H) \varepsilon_{h_k}(\lambda_i), \\ & \text{for } j = i, \dots, i + m - 1. \end{aligned} \quad (3.20)$$

Combining with the error estimate

$$\begin{aligned} \|\mathcal{R}_j^T \mathcal{P}^{-1} U - P_{h_{k+1}} \mathcal{R}_j^T \mathcal{P}^{-1} U\|_1 & \lesssim \delta_{h_{k+1}}(\lambda_i), \\ & \text{for } j = i, \dots, i + m - 1, \end{aligned} \quad (3.21)$$

we have

$$\|\tilde{u}_{j,h_{k+1}} - \mathcal{R}_j^T \mathcal{P}^{-1} U\|_1 \lesssim \eta_a^*(H) \varepsilon_{h_k}(\lambda_i) + \delta_{h_{k+1}}(\lambda_i),$$

$$\text{for } j = i, \dots, i + m - 1. \quad (3.22)$$

After Step 3, from the definition of $V_{H,h_{k+1}}$ and (3.22), we derive

$$\begin{aligned} & \sup_{u \in M(\lambda_i), \|u\|_1=1} \inf_{v_{H,h_{k+1}} \in V_{H,h_{k+1}}} \|u - v_{H,h_{k+1}}\|_1 \\ & \leq \sup_{u \in M(\lambda_i), \|u\|_1=1} \inf_{v_{h_{k+1}} \in W_{h_{k+1}}} \|u - v_{h_{k+1}}\|_1 \\ & \lesssim \sup_{v_{h_{k+1}} \in W_{h_{k+1}}, \|v_{h_{k+1}}\|_1=1} \inf_{u \in M(\lambda_i)} \|v_{h_{k+1}} - u\|_1 \\ & \lesssim \max_{j=i, \dots, i+m-1} \|\tilde{u}_{j,h_{k+1}} - \mathcal{R}_j^T \mathcal{P}^{-1} U\|_1 \\ & \lesssim \eta_a^*(H) \varepsilon_{h_k}(\lambda_i) + \delta_{h_{k+1}}(\lambda_i), \end{aligned} \quad (3.23)$$

where $W_{h_{k+1}} := \text{span}\{\tilde{u}_i^{h_{k+1}}, \dots, \tilde{u}_{i+m-1}^{h_{k+1}}\}$.

Similarly,

$$\begin{aligned} & \sup_{u_* \in M^*(\lambda), \|u\|_1=1} \inf_{v_{H,h_{k+1}} \in V_{H,h_{k+1}}^*} \|u^* - v_{H,h_{k+1}}\|_1 \\ & \lesssim \eta_a(H) \varepsilon_{h_k}^*(\lambda_i) + \delta_{h_{k+1}}^*(\lambda_i). \end{aligned} \quad (3.24)$$

Then from the error estimate results stated in Theorem 2.2 for the eigenvalue problem (see, e.g., [3, Section 8]) and (3.23)-(3.24), the following error estimates hold

$$\Theta(M(\lambda_i), M_{h_{k+1}}(\lambda_i)) \lesssim \eta_a^*(H) \varepsilon_{h_k}(\lambda_i) + \delta_{h_{k+1}}(\lambda_i), \quad (3.25)$$

$$\Theta(M^*(\lambda_i), M_{h_{k+1}}^*(\lambda_i)) \lesssim \eta_a(H) \varepsilon_{h_k}^*(\lambda_i) + \delta_{h_{k+1}}^*(\lambda_i). \quad (3.26)$$

These are the desired estimates (3.11) and (3.12). Furthermore,

$$\begin{aligned} \Phi(M(\lambda_i), M_{h_{k+1}}(\lambda_i)) & \lesssim \tilde{\eta}_a^*(H) \sup_{u \in M(\lambda), \|u\|_1=1} \inf_{v_{H,h_{k+1}} \in V_{H,h_{k+1}}} \|u - v_{H,h_{k+1}}\|_1 \\ & \leq \eta_a^*(H) \varepsilon_{h_{k+1}}(\lambda_i), \end{aligned} \quad (3.27)$$

where

$$\tilde{\eta}_a^*(H) := \sup_{f \in V, \|f\|_b=1} \inf_{v_{H,h_{k+1}} \in V_{H,h_{k+1}}} \|T_* f - v_{H,h_{k+1}}\|_1 \leq \eta_a^*(H). \quad (3.28)$$

Then we obtain (3.13). A similar argument leads to (3.14). \square

4. Multilevel correction scheme. In this section, we introduce a multilevel correction scheme based on the *One Correction Step* 3.1. The method improves accuracy after each correction step, which is different from the two-grid methods in [9, 23, 25].

ALGORITHM 4.1. *Multilevel Correction Scheme*

1. Construct a coarse conforming finite element space V_{h_1} on \mathcal{T}_{h_1} such that $V_H \subset V_{h_1}$ and solve the following two eigenvalue problems:

Find $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1}$ such that $b(u_{h_1}, u_{h_1}) = 1$ and

$$a(u_{h_1}, v_{h_1}) = \lambda_{h_1} b(u_{h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}. \quad (4.1)$$

Find $(\lambda_{h_1}, u_{h_1}^*) \in \mathcal{R} \times V_{h_1}$ such that $b(u_{h_1}^*, u_{h_1}^*) = 1$ and

$$a(v_{h_1}, u_{h_1}^*) = \lambda_{h_1} b(v_{h_1}, u_{h_1}^*), \quad \forall v_{h_1} \in V_{h_1}. \quad (4.2)$$

Choose $2q$ eigenpairs $\{\lambda_{j,h_1}, u_{j,h_1}\}_{j=i}^{i+q-1}$ and $\{\lambda_{j,h_1}, u_{j,h_1}^*\}_{j=i}^{i+q-1}$ which approximate the desired eigenvalue λ_i and its geometric eigenspaces of the eigenvalue problem (4.1) and its adjoint one (4.2). Based on these two geometric eigenspace, we compute the corresponding algebraic eigenspaces $M_{h_1}(\lambda_i) := \text{space}\{u_{i,h_1}, \dots, u_{i+m-1,h_1}\}$ and $M_{h_1}^*(\lambda_i) := \text{space}\{u_{i,h_1}^*, \dots, u_{i+m-1,h_1}^*\}$. Then do the following correction steps.

2. Construct a series of finer finite element spaces V_{h_2}, \dots, V_{h_n} on the sequence of nested meshes $\mathcal{T}_{h_2}, \dots, \mathcal{T}_{h_n}$ (cf. [4, 6]).
3. Do $k = 1, \dots, n-1$
Obtain new eigenpair approximations $(\{\lambda_{j,h_{k+1}}\}_{j=i}^{i+m-1}, M_{h_{k+1}}(\lambda_i), M_{h_{k+1}}^*(\lambda_i))$ by Algorithm 3.1

$$\begin{aligned} & (\{\lambda_{j,h_{k+1}}\}_{j=i}^{i+m-1}, M_{h_{k+1}}(\lambda_i), M_{h_{k+1}}^*(\lambda_i)) = \\ & \text{Correction}(V_H, \{\lambda_{j,h_k}\}_{j=i}^{i+m-1}, M_{h_k}(\lambda_i), M_{h_k}^*(\lambda_i), V_{h_{k+1}}). \end{aligned}$$

End Do

Finally, we obtain eigenpair approximations $(\{\lambda_{j,h_n}\}_{j=i}^{i+m-1}, M_{h_n}(\lambda_i), M_{h_n}^*(\lambda_i))$.

THEOREM 4.1. After implementing Algorithm 4.1, the resultant eigenpair approximation $(\{\lambda_{j,h_n}\}_{j=i}^{i+m-1}, M_{h_n}(\lambda_i), M_{h_n}^*(\lambda_i))$ has following error estimates

$$\Theta(M(\lambda_i), M_{h_n}(\lambda_i)) \lesssim \varepsilon_{h_n}(\lambda_i), \quad (4.3)$$

$$\Phi(M(\lambda_i), M_{h_n}(\lambda_i)) \lesssim \eta_a^*(H) \varepsilon_{h_n}(\lambda_i), \quad (4.4)$$

$$\Theta(M^*(\lambda_i), M_{h_n}^*(\lambda_i)) \lesssim \varepsilon_{h_n}^*(\lambda_i), \quad (4.5)$$

$$\Phi(M^*(\lambda_i), M_{h_n}^*(\lambda_i)) \lesssim \eta_a(H) \varepsilon_{h_n}^*(\lambda_i), \quad (4.6)$$

$$|\widehat{\lambda}_{i,h_n} - \lambda_i| \lesssim \varepsilon_{h_n}(\lambda_i) \varepsilon_{h_n}^*(\lambda_i), \quad (4.7)$$

where $\widehat{\lambda}_{i,h_n} = \frac{1}{m} \sum_{j=i}^{i+m-1} \lambda_{j,h_n}$, $\varepsilon_{h_n}(\lambda_i) = \sum_{k=1}^n \eta_a^*(H)^{n-k} \delta_{h_k}(\lambda_i)$ and $\varepsilon_{h_n}^*(\lambda_i) = \sum_{k=1}^n \eta_a(H)^{n-k} \delta_{h_k}^*(\lambda_i)$.

Proof. First, the following estimates hold

$$\Theta(M(\lambda_i), M_{h_1}(\lambda_i)) \lesssim \varepsilon_{h_1}(\lambda_i), \quad (4.8)$$

$$\Phi(M(\lambda_i), M_{h_1}(\lambda_i)) \lesssim \eta_a^*(h_1) \varepsilon_{h_1}(\lambda_i) \leq \eta_a^*(H) \varepsilon_{h_1}(\lambda_i), \quad (4.9)$$

$$\Theta(M^*(\lambda_i), M_{h_1}^*(\lambda_i)) \lesssim \varepsilon_{h_1}^*(\lambda_i), \quad (4.10)$$

$$\Phi(M^*(\lambda_i), M_{h_1}^*(\lambda_i)) \lesssim \eta_a(h_1) \varepsilon_{h_1}^*(\lambda_i) \leq \eta_a(H) \varepsilon_{h_1}^*(\lambda_i). \quad (4.11)$$

Then we set $\varepsilon_{h_1}(\lambda_i) := \delta_{h_1}(\lambda_i)$ and $\varepsilon_{h_1}^*(\lambda_i) := \delta_{h_1}^*(\lambda_i)$.

By recursive relation and Theorem 3.1, we derive

$$\begin{aligned} \Theta(M(\lambda_i), M_{h_n}(\lambda_i)) & \lesssim \varepsilon_{h_n}(\lambda_i) = \eta_a^*(H) \varepsilon_{h_{n-1}}(\lambda_i) + \delta_{h_n}(\lambda_i) \\ & \lesssim \eta_a^*(H)^2 \varepsilon_{h_{n-2}}(\lambda_i) + \eta_a^*(H) \delta_{h_{n-1}}(\lambda_i) + \delta_{h_n}(\lambda_i) \\ & \lesssim \sum_{k=1}^n \eta_a^*(H)^{n-k} \delta_{h_k}(\lambda_i) \end{aligned} \quad (4.12)$$

and

$$\Phi(M(\lambda_i), M_{h_n}(\lambda_i)) \lesssim \eta_a^*(H) \sum_{k=1}^n \eta_a^*(H)^{n-k} \delta_{h_k}(\lambda_i). \quad (4.13)$$

These are the estimates (4.3) and (4.4) and the estimates (4.5) and (4.6) can be proved similarly. From Theorem 2.2, (4.3) and (4.5), we can obtain the estimate (4.7). \square

5. Numerical results. In this section, we give some numerical results to illustrate the efficiency of the multilevel correction scheme defined by Algorithm 4.1. Here, we solve the following eigenvalue problem

$$\begin{cases} -\Delta u + \mathbf{b} \cdot \nabla u &= \lambda u, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

where $\mathbf{b} = [b_1, b_2]^T \in \mathcal{C}^2$ is a constant vector and $\Omega = (0, 1) \times (0, 1)$. This example comes from [7, 8]. We choose $b_1 = 1$ and $b_2 = 1/2$ in Subsections 5.1 and 5.2. Then we choose $b_1 = \cos(\pi x_1) \sin(\pi x_2)$ and $b_2 = -\sin(\pi x_1) \cos(\pi x_2)$ in Subsection 5.3. We also choose a complex vector \mathbf{b} in the final example.

When $b_1 = 1$ and $b_2 = 1/2$, the problem (5.1) is nonself-adjoint, but all of its eigenvalues are nondefective (all algebraic eigenfunctions are of order 1) and real numbers

$$\lambda_{k,\ell} = \frac{b_1^2 + b_2^2}{4} + (k^2 + \ell^2)\pi^2, \quad (5.2)$$

for $k, \ell \in \mathcal{N}^+$.

The corresponding eigenfunctions can be chosen as real functions

$$u_{k,\ell} = \exp\left(\frac{b_1 x_1 + b_2 x_2}{2}\right) \sin(k\pi x_1) \sin(\ell\pi x_2). \quad (5.3)$$

The corresponding adjoint eigenvalue problem has eigenvalues $\lambda_{k,\ell}$ and eigenfunctions

$$u_{k,\ell}^* = \exp\left(-\frac{b_1 x_1 + b_2 x_2}{2}\right) \sin(k\pi x_1) \sin(\ell\pi x_2). \quad (5.4)$$

5.1. Multi-space way. In this case, finer finite element spaces are constructed by increasing polynomial degrees of the beginning finite element space on the same mesh. We first solve the eigenvalue problem (2.12) by linear finite element on a relatively coarser mesh \mathcal{T}_H , then perform the first correction step with quadratic element, followed by cubic element for the second correction step and quartic element for the third correction step. Our initial mesh \mathcal{T}_H is obtained from the Delaunay triangulation followed by four levels of regular mesh refinement. Figure 5.1 depicts errors for the first eigenvalue $(5/16 + 2\pi^2)$ approximation, and Figure 5.2 plots numerical errors for the eigenfunction and the corresponding adjoint eigenfunction associated with the first eigenvalue.

Furthermore, Figure 5.3 provides numerical results for the summation of the errors for the first 6 eigenvalues: $5/16 + [2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2, 10\pi^2]$.

From Figures 5.1-5.3, we find that each correction step improves the convergence order by two for eigenvalue approximation, and by one for eigenfunction approximation when the exact eigenfunction is sufficiently smooth.

To end this subsection, we make a comparison with the PPR method [13]. We see from Figure 5.4 that the two-level correction scheme by the multi-space way has slightly better accuracy than the PPR method. However, the two-level correction needs to solve two extra boundary value problems while the PPR method only need to perform a local recovery at each node. Thus, we should say that the PPR method has better efficiency than the two-level correction under regular mesh refinement when the eigenfunction has regularity $H^3(\Omega) \cap W^{2,\infty}(\Omega)$. Nevertheless, three and four-level correction will outperform the PPR method.

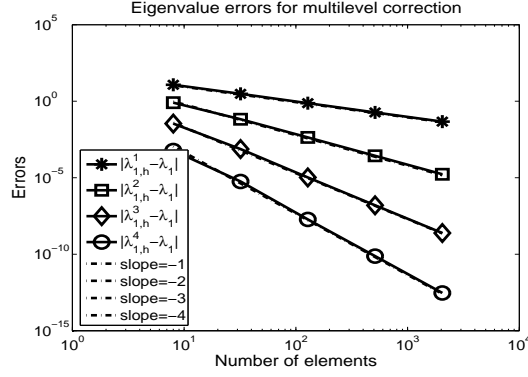


FIG. 5.1. Here, λ_h^1 denote the eigenvalue approximation by linear element, λ_h^2 is the eigenvalue approximation by the first correction with quadratic element, λ_h^3 the eigenvalue approximation by the second correction with cubic element, λ_h^4 the eigenvalue approximation by the third correction with quartic element

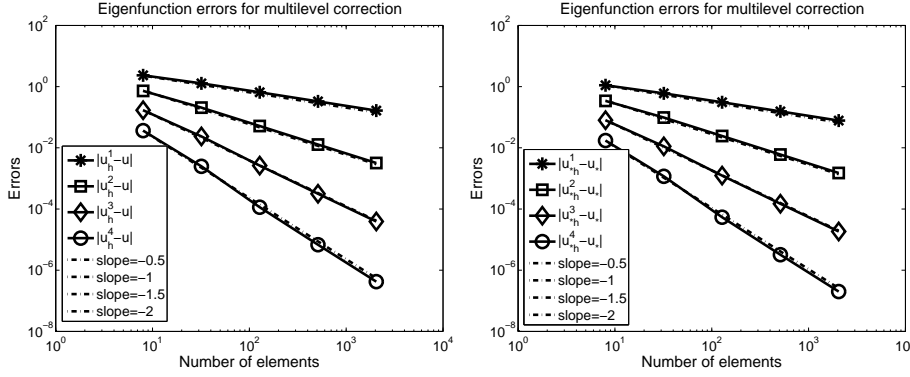


FIG. 5.2. Here, u_h^1 and u_{*h}^1 denote the eigenfunction approximation and its adjoint approximation by linear element, u_h^2 and u_{*h}^2 are eigenfunction approximation and its adjoint approximation by the first correction with quadratic element, u_h^3 and u_{*h}^3 , eigenfunction and its adjoint approximation by the second correction with cubic element, u_h^4 and u_{*h}^4 , eigenfunction and its adjoint approximation by the third correction with quartic element

5.2. Multi-grid way. An alternative way of the multilevel correction scheme is to construct finer finite element spaces by mesh refinement. We first solve the eigenvalue problem (2.12) in the linear finite element space on an initial coarse mesh \mathcal{T}_H ($\mathcal{T}_{h_1} := \mathcal{T}_H$). Then we refine the mesh regularly with the resultant meshes \mathcal{T}_{h_k} satisfying $h_k = 2^{1-k}H$ for $(k = 2, \dots, n)$, and solve auxiliary source problems (3.1) and (3.2) in the linear finite element space V_{h_k} defined on \mathcal{T}_{h_k} and the corresponding eigenvalue problems (3.3) and (3.4) in V_{H,h_k} . We have the following estimate

$$\varepsilon_{h_n}(\lambda) = \sum_{k=1}^n H^{n-k} h_k = \sum_{k=1}^n (2H)^{n-k} h_n \leq \frac{1}{1-2H} h_n \approx h_n,$$

and similarly $\varepsilon_{h_n}^*(\lambda) \approx h_n$, which implies that the multilevel correction method achieves the optimal convergence rate if the initial mesh size H is reasonably small, say $H = 1/4$ as we will use in our numerical tests.

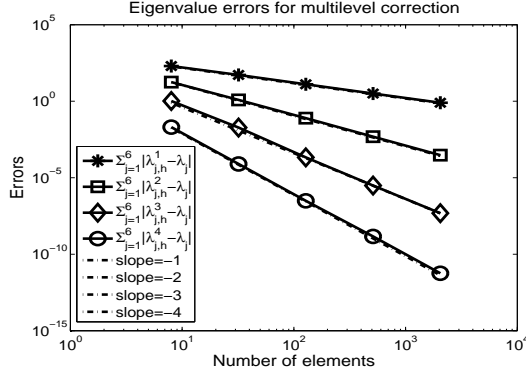


FIG. 5.3. Approximation errors for the summation of the errors for the first 6 eigenvalues by the multi-space way. Here, $\lambda_{j,h}^1$ denote the eigenvalue approximation by linear element, $\lambda_{j,h}^2$ is the eigenvalue approximation by the first correction with quadratic element, $\lambda_{j,h}^3$ the eigenvalue approximation by the second correction with cubic element, $\lambda_{j,h}^4$ the eigenvalue approximation by the third correction with quartic element

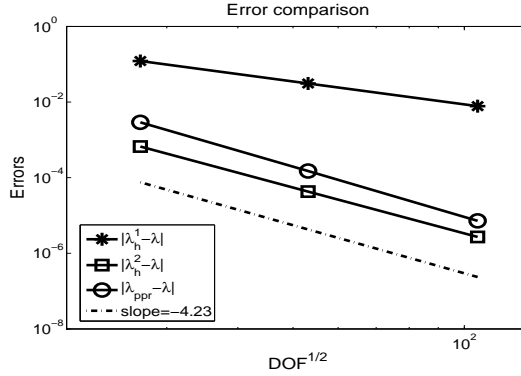


FIG. 5.4. Comparison with the PPR method in [13] when $b_1 = 10$ and $b_2 = 1$. λ_h^1 , eigenvalue approximation by linear element; λ_h^2 , eigenvalue approximation by the first correction with quadratic element; λ_{ppr} , the eigenvalue approximation by the PPR method

Numerical results for the first eigenvalue $\lambda = 5/16 + 2\pi^2$ and the two associated eigenfunctions are demonstrated in Figures 5.5 and 5.6, respectively. Here we use the uniform meshes with $H = 1/4$. Furthermore, Figure 5.7 provides numerical results for the summation of the errors for the first 6 eigenvalues: $5/16 + [2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2, 10\pi^2]$ with $H = 1/8$ and $H = 1/16$, respectively.

We observe from Figures 5.5-5.7, that our multilevel correction method with the multi-grid way produces eigenvalue and eigenfunction approximations with the optimal convergence rate. Therefore, we can combine the multigrid method for boundary value problems and our multilevel correction scheme (cf. [11, 18]) to achieve better efficiency for nonsymmetric eigenvalue problems.

5.3. Eigenvalue problem on L -shape domain. In this subsection, we consider the eigenvalue problem (5.1) on the L -shape domain $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1) \times (-1, 0]$. Since Ω has a reentrant corner, the singularity of eigenfunctions is expected. As a consequence, the convergence rate for the first eigenvalue approximation is $4/3$ by

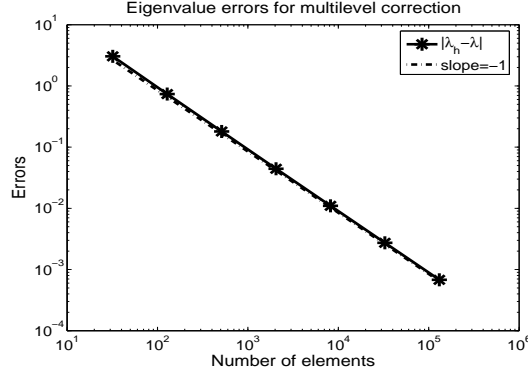


FIG. 5.5. Approximation errors for the first eigenvalue $5/16 + 2\pi^2$ by the multi-grid way with $H = 1/4$

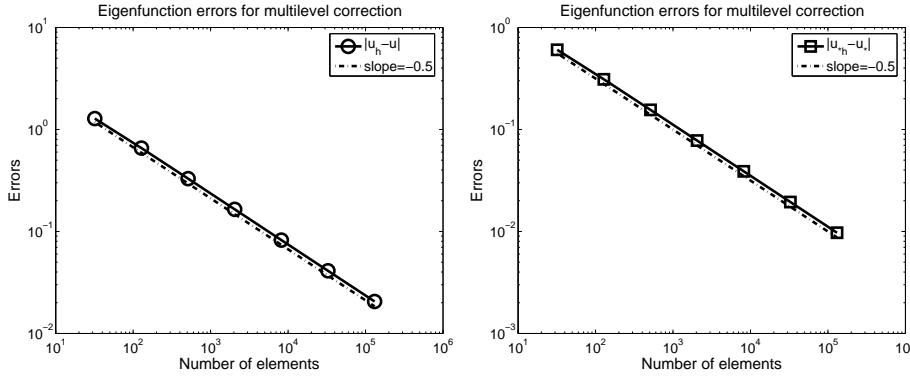


FIG. 5.6. Approximation errors for the first eigenfunction and its adjoint by the multi-grid way with $H = 1/4$

the linear finite element method on quasi-uniform meshes. Since the exact eigenvalue is unknown, we choose an adequately accurate approximation $\lambda = 9.95240442893276$ as the exact first eigenvalue for our numerical tests.

Our multilevel correction scheme is tested on a sequence of meshes \mathcal{T}_H ($\mathcal{T}_{h_1} := \mathcal{T}_H$), $\mathcal{T}_{h_2}, \dots, \mathcal{T}_{h_n}$ produced by the adaptive refinement (cf. [17, 24]). Here the ZZ recovery method (cf. [26]) is adopted as the *a posteriori* error estimator for eigenfunction and adjoint eigenfunction approximations $\sqrt{\|u_h - u\|_{a,h}^2 + \|u_h^* - u^*\|_{a,h}^2}$. Figure 5.8 shows the initial mesh and the mesh after 12 adaptive iterations. Figure 5.9 gives the corresponding numerical results for the adaptive iterations.

From Figure 5.9, we observe that the multilevel correction method works well on adaptive meshes with the optimal convergence rate. Furthermore, the situation is very different from the two-grid [9, 23, 25] method in that the initial mesh has very little impact on the finest one. Thus the multilevel correction method can be coupled with the adaptive refinement naturally.

5.4. Eigenvalue problem with complex vector. In this subsection, we test the multilevel correction scheme for the problem (5.1) with complex vector $\mathbf{b} = [1 + 2i, 1/2 - i]^T$. We use the multi-space and multi-grid ways as in Subsections 5.1 and

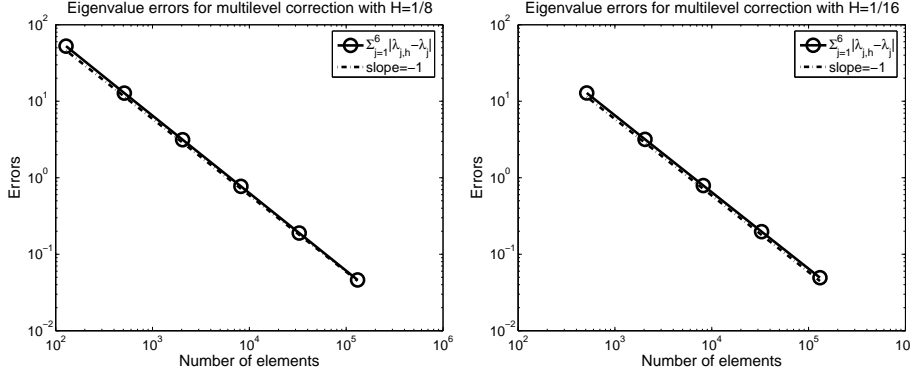


FIG. 5.7. Approximation errors for the error summation of the first 6 eigenvalues by the multigrid way with $H = 1/8$ (left) and $1/16$ (right)

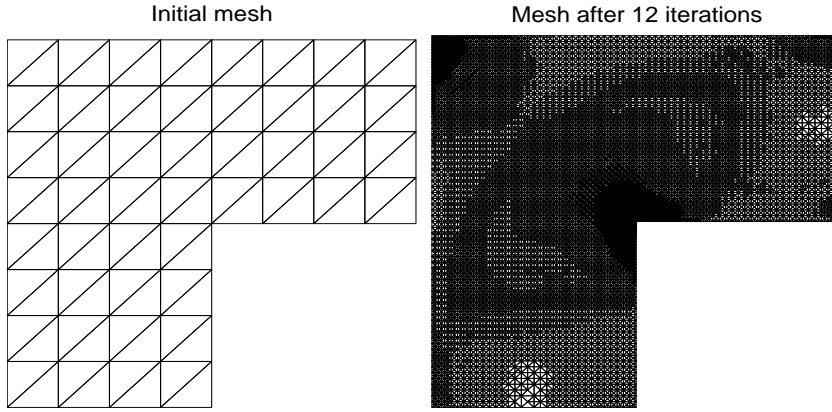


FIG. 5.8. The initial mesh and the one after 12 adaptive iterations for the L-shape domain

5.2, respectively, to check the multilevel correction scheme. Figure 5.10 shows the numerical results for the first 6 eigenvalues. It is observed from Figure 5.10 that the multilevel correction method defined in Algorithm 4.1 can also work very well for the nonsymmetric eigenvalue problems with complex vector.

6. Concluding remarks. In this paper, we propose and analyze a multilevel correction scheme to improve the efficiency of both defective and nondefective nonsymmetric eigenpair approximations. In this multilevel correction, we only need to solve eigenvalue problems in the coarsest finite element space. Sometimes, we also need to compute the algebraic eigenspace based on the geometric eigenspace when the ascent is larger than 1.

Furthermore, our multilevel correction scheme can be coupled with the multigrid method to construct a parallel method for eigenvalue problems (see, e.g., [11, 12, 18, 24]). It can also be combined with adaptive techniques (cf. [17]) for singular eigenfunction cases. These will be our future work.

A final remark. As long as higher eigenvalues are concerned, the multi-space way is preferred (than the multi-grid way). We can see it clearly by comparing numerical accuracies for summations of the first 6 eigenvalues in §5.1 and §5.2.

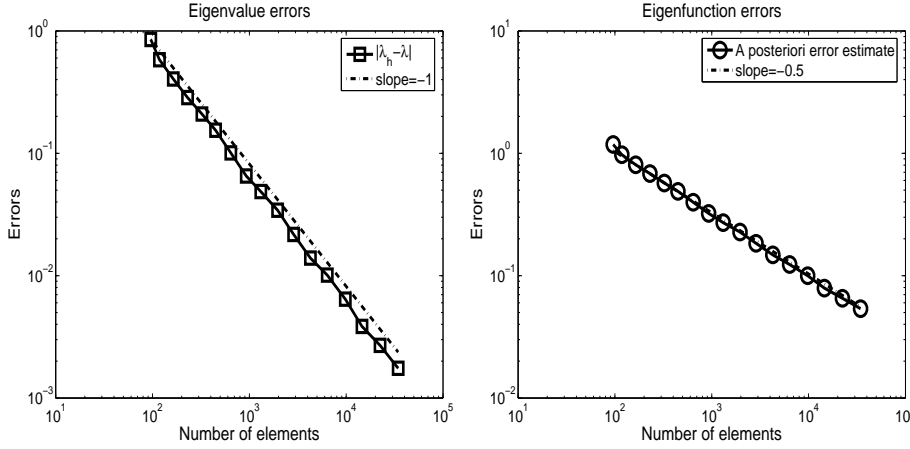


FIG. 5.9. Approximation errors of the first eigenvalue and the a posteriori errors of the associated eigenfunction and adjoint eigenfunction

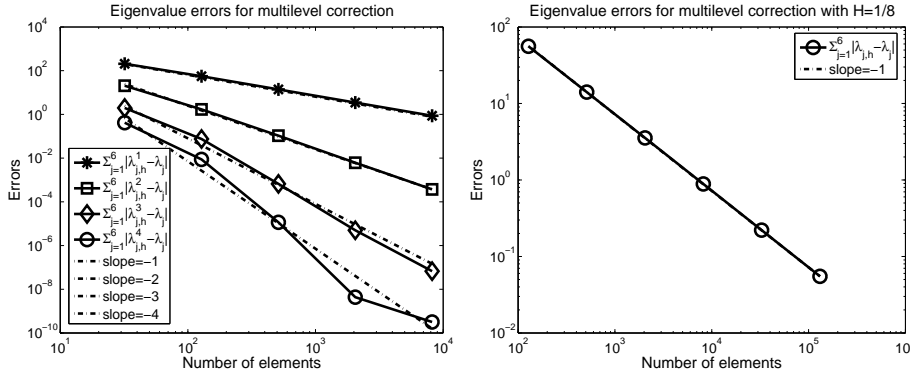


FIG. 5.10. Approximation errors for the summation of errors for the first 6 eigenvalues by the multi-space way (left) and the multi-grid way with $H = 1/8$ (right). Here, $\lambda_{j,h}^1$ denote the eigenvalue approximation by linear element, $\lambda_{j,h}^2$ is the eigenvalue approximation by the first correction with quadratic element, $\lambda_{j,h}^3$ the eigenvalue approximation by the second correction with cubic element, $\lambda_{j,h}^4$ the eigenvalue approximation by the third correction with quartic element

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REFERENCES

- [1] K. A. Cliffe, E. J. C. Hall and P. Houston, Adaptive discontinuous Galerkin methods for eigenvalue problems arising in incompressible fluid flows, SIAM J. Sci. Comput., 31(6) (2010), pp. 4607-4632.

- [2] I. Babuška and J. E. Osborn, Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems, *Math. Comp.* 52 (1989), pp. 275-297.
- [3] I. Babuška and J. Osborn, Eigenvalue Problems, In *Handbook of Numerical Analysis*, Vol. II, (Eds. P. G. Lions and Ciarlet P.G.), Finite Element Methods (Part 1), North-Holland, Amsterdam, pp. 641-787, 1991.
- [4] S. Brenner and L. Scott, *The Mathematical Theory of Finite Element Methods*, New York: Springer-Verlag, 1994.
- [5] F. Chatelin, *Spectral Approximation of Linear Operators*, Academic Press Inc, New York, 1983.
- [6] P. G. Ciarlet, *The finite Element Method for Elliptic Problem*, North-holland Amsterdam, 1978.
- [7] V. Heuveline and R. Rannacher, A posteriori error control for finite element approximations of elliptic eigenvalue problems, *Adv. Comput. Math.*, 15 (2001), pp. 107-138.
- [8] V. Heuveline and R. Rannacher, Adaptive FEM for eigenvalue problems with application in hydrodynamic stability analysis, *J. Numer. Math.*, submitted, 2006.
- [9] K. Kolman, A two-level method for Nonsymmetric eigenvalue problems, *Acta Mathematicae Applicatae Sinica (English series)*, 2(1) (2005), pp. 1-12.
- [10] Q. Lin, F. Luo and H. Xie, *A multilevel correction method for Stokes eigenvalue problems and its applications*, *Math. Methods Appl. Sci.*, DOI: 10.1002/mma.2866, 2013.
- [11] Q. Lin and H. Xie, *A multi-level correction scheme for eigenvalue problems*, *Math. Comp.*, DOI: <http://dx.doi.org/10.1090/S0025-5718-2014-02825-1>, 2014.
- [12] Q. Lin and H. Xie, *A type of multigrid method for eigenvalue problem*, Research Report in ICMSEC, 2011-06 (2011).
- [13] A. Naga and Z. Zhang, *Function value recovery and its application in eigenvalue problems*, *SIAM J. Numer. Anal.*, 50(1) (2012), pp. 272-286.
- [14] A. Naga, Z. Zhang and A. Zhou, Enhancing eigenvalue approximation by gradient recovery, *SIAM J. Sci. Comput.*, 28(4) (2006), pp. 1289-1300.
- [15] P. Schmid and D. S. Henningson, *Stability and Transition in Shear Flows*, Appl. Math. Sci. 142, Springer, New York, 2001.
- [16] V. Shaidurov, *Multigrid methods for finite element*, Kluwer Academic Publics, Netherlands, 1995.
- [17] H. Wu and Z. Zhang, Enhancing eigenvalue approximation by gradient recovery on adaptive meshes, *IMA J. Numer. Anal.*, 29(4) (2009), pp. 1008-1022.
- [18] H. Xie, A type of multilevel method for the Steklov eigenvalue problem, *IMA J. Numer. Anal.*, 1-17, doi:10.1093/imanum/drt009, 2013.
- [19] H. Xie, A multigrid method for eigenvalue problem, *J. Comput. Phys.*, 274 (2014), 550-561.
- [20] J. Xu, Iterative methods by space decomposition and subspace correction, *SIAM Review*, 34(4) (1992), pp. 581-613.
- [21] J. Xu, A new class of iterative methods for nonselfadjoint or indefinite problems, *SIAM J. Numer. Anal.*, 29 (1992), pp. 303-319.
- [22] J. Xu, A novel two-grid method for semilinear elliptic equations, *SIAM J. Sci. Comput.*, 15 (1994), pp. 231-237.
- [23] J. Xu and A. Zhou, A two-grid discretization scheme for eigenvalue problems, *Math. Comput.*, 70(233) (2001), 17-25.
- [24] J. Xu and A. Zhou, Local and parallel finite element algorithm for eigenvalue problems, *Acta Math. App. l. Sin. Engl. Ser.*, 18(2) (2002), pp. 185-200.
- [25] Y. Yang and X. Fan, Generalized Rayleigh quotient and finite element two-grid discretization schemes, *Science in China Series A: mathematics*, 52(9) (2009), pp. 1955-1972.
- [26] O. Zienkiewicz and J. Zhu, A simple error estimator and adaptive procedure for practical engineering analysis, *Int. J. Numer. Methods Eng.*, 24 (1987), pp. 337-357.